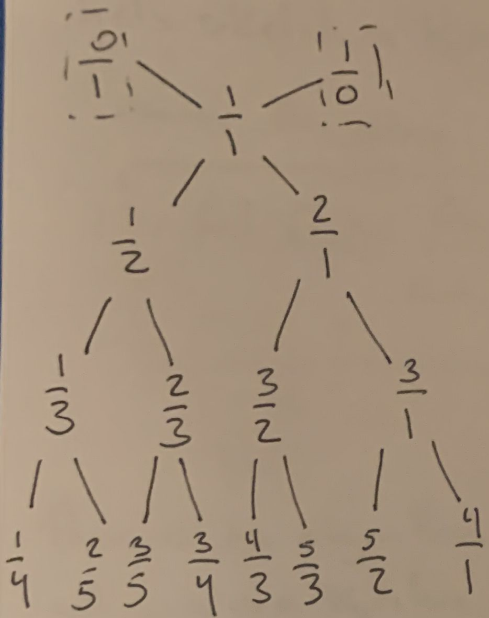


Lets define a seq using SB Tree
Stern's <u>Diatomic</u> seq/series

Let $a_0 = 0$, $a_1 = 1$, $a_{2n} = a_n$, $a_{2n+1} = a_n + a_{n+1}$

Lets check
$a_1 = 1$
$a_2 = a_1 = 1$
$a_3 = a_1 + a_2 = 1+1 = 2$
$a_4 = a_2 = 1$
$a_5 = a_2 + a_3 = 1+2 = 3$
$a_6 = a_3 = 2$
$a_7 = a_3 + a_4 = 2+1 = 5$
$a_8 = a_4 = 1$

For two fractions $\frac{a}{b}$ and $\frac{c}{d}$ their mediant is $\frac{a+c}{b+d}$

So $\{a_k\} = 0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1 \ldots$

We can visualize this series with a 2D array (Stern's Diatomic Array)

1 1
1 2 1
1 3 2 3 1
1 4 3 5 2 5 3 4 1
1 5 4 3 8 5 7 2 7 5 8 3 7 4 5 1

a_1 a_2
a_2 a_3 a_4
a_4 a_5 a_6 a_7 a_8
a_8 a_9 a_{10} a_{11} a_{12} a_{13} a_{14} a_{15} a_{16}

First row has two terms after zero in SDS (a_1, a_2)
Given n^{th} row, generate $n+1^{th}$ by copying, but inserting new term between two terms that is their sum.

Similar to Pascal's Triangle (we will define when we introduce the Binomial Theorem). Each row is palindrome, also

<u>Claim:</u> $a_{2^n+k} = a_k + a_{2^n-k}$;

Lets sketch a bijection from $\mathbb{N}$ to $\mathbb{Q}^+$; $f: \mathbb{N} \to \mathbb{Q}^+$

using the series

Let $f(n) = \frac{a_n}{a_{n+1}}$

⇑

Wait.

There is an algorithm that will give us a bijection from $\mathbb{N} \to \mathbb{Q}^+$

Using the SB tree.

It is an infinite binary tree

(actually its an infinite binary search tree)

So what tree traversal algorithms have you heard of?

$$f(1) = \frac{a_1}{a_{1+1}} = \frac{1}{1}$$

$$f(2) = \frac{a_2}{a_{2+1}} = \frac{1}{2}$$

$$f(3) = \frac{a_3}{a_{3+1}} = \frac{2}{1}$$

$$f(4) = \frac{a_4}{a_{4+1}} = \frac{1}{3}$$

$$f(5) = \frac{a_5}{a_{5+1}} = \frac{3}{2}$$

- $f(n) = a_n / a_{n+1}$

We can now say that the 5^{th} rational number is $2/3$.

Claim $f: \mathbb{N} \to \mathbb{Q}^+$ is a bijection [show this in HW]

Since f a bijection, f^{-1} exists. Given rational, we can find natural

<u>Ex</u> Given $q = 1/4$, Can we find n s.t. $f(n) = 1/4$?

<u>Verify on HW</u>

$$\begin{cases} f^{-1}(1) = 1 \\ f^{-1}(q) = 2f^{-1}\left(\frac{q}{1-q}\right) & \text{if } q < 1 \\ f^{-1}(q) = 2f^{-1}(q-1) + 1 & \text{if } q > 1 \end{cases}$$

<u>Ex:</u>

$$f^{-1}\left(\frac{3}{2}\right) = 2f^{-1}\left(\frac{3/2}{1-3/2}\right) + 1 = 2f^{-1}\left(\frac{1}{2}\right) + 1 = 5$$

Now that we have a bijection from $\mathbb{N} \to \mathbb{Q}^+$, we can extend to $\mathbb{N} \to \mathbb{Q}$ by using $\mathbb{N} \to \mathbb{Z}$ and writing $\mathbb{Z} \to \mathbb{Q}$

Our bijection from

$\mathbb{N} \to \mathbb{E}$ $\qquad$ $\mathbb{N} \to \mathbb{Z}$

0, 1, 2 $\quad$ 0, 2, 4

$$n \to \begin{cases} -n & \text{even} \\ n+1 & \text{odd} \end{cases} \qquad n \to \begin{cases} n/2 & \text{even} \\ \frac{-(n+1)}{2} & \text{odd} \end{cases}$$

Take
$$g(z) = \begin{cases} \frac{a_z}{a_{z+1}} & \text{if } z > 0 \\ \frac{-a_z}{a_{-(z-1)}} & \text{if } z < 0 \\ 0 & \text{if } z = 0 \end{cases}$$

This is a bijection from $\mathbb{Z} \to \mathbb{Q}$

We can write $\mathbb{Q} \to \mathbb{N}$ by using function composition

take inverse of g, g^{-1}:

$$g^{-1}(q) = \begin{cases} 2f^{-1}(q-1)+1 & \text{if } q > 1 \\ 1 & q = 1 \\ 2f^{-1}\left(\frac{q}{1-q}\right) & \text{if } 0 < q < 1 \\ 0 & \text{if } q = 0 \\ -2\left(f^{-1}\left(\frac{-q}{1+q}\right)\right) & -1 < q < 0 \\ -1 & q = -1 \\ -2\left(\left(f^{-1}(-q-1\right)\right)+1 & \text{if } q < -1 \end{cases}$$

Define inverse

$$h^{-1}(z) = \begin{cases} 2z & z > 0 \\ 1 & z = 0 \\ -2z-1 & z < 0 \end{cases}$$

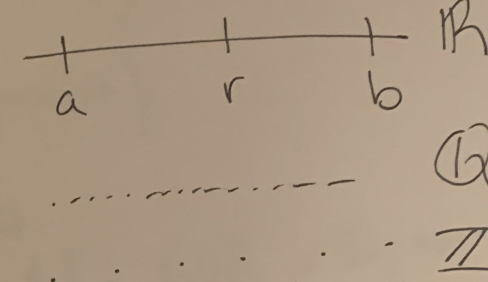
Then $h^{-1} \circ g^{-1} : \mathbb{Q} \to \mathbb{N}$

Thm Given any two numbers $a, b \in \mathbb{R}$, $\exists\, r \in \mathbb{Q}$ s.t

$a < r < b$ Rationals are Dense in $\mathbb{R}$

Even though $\mathbb{Q}$ is not all of $\mathbb{R}$, it kind of fills up space b/t them

$\Rightarrow$ Implication is that every real number $(\pi, e, \sqrt{2})$ can be approximated by rational numbers



Proof

Suppose $a < b$, we want to show

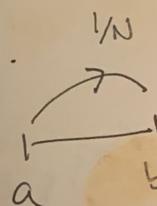
$\exists\, r$ s.t $r = \frac{M}{N}$ and $a < \frac{M}{N} < r$

1) Look at denominator

We know $a < b$ so $b - a > 0$ (small positive number that is our unit of measurement)

Def (Archimedean Property)

For, Every positive rational $\frac{M}{N}$ where $M, N \in \mathbb{Z}^+$, if we add more than N copies, the resulting sum is more than 1.

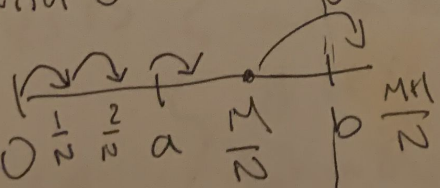


$\Rightarrow \exists\, N \in \mathbb{Z}^+$ s.t $N(b-a) > 1 \Rightarrow b - a > \frac{1}{N}$

This says that even though b and a are close, they're actually not that close. Therefore we found the denominator

2) Look at numerator (assume $b > a > 0$, for the other cases we'll adapt by flipping inequality and negative signs)

Idea. Start with 0 remembering N is fixed.



Successively add 1/N until we reach last number $\frac{M}{N}$ that is less than b.

i.e. $\frac{M+1}{N} > b$

It turns out that the last element is what solves the problem

$\left(r = \frac{M}{N}\right)$

Let $S = \{ \frac{M}{N}$ s.t. $M = 0, 1, 2, 3, \ldots, \frac{M}{N} < b \}$

S is non-empty b/c $0 \in S$ (b is positive)

by construction S is bounded above by b.

We'll use this idea about real numbers called a LUB or Supremum.

We'll use this idea about real numbers called a LUB or supremum, with one caveat.

$\Rightarrow$ Intuitively think of it like a MAXIMUM, with one caveat.

Lets say you did not get best grade in class. You think who got a better grade than me. If something is not a supremum, something else is bigger than it.

~~Let X be ... bound of S. Then X = Sup(X) ...~~

If X a partially ordered set and S a subset then S_0 is $Sup(S)$

iff

1) $s \leq S_0 \ \forall s \in S$

2) if $t \in X$ s.t. $s \leq t \ \forall s \in S$ then $S_0 \leq t$

but also $\forall a \in \mathbb{R}_-$

if $a \leq b \ \forall a \in \mathbb{R}_-$ then $0 \leq b$

Max

M is max of S iff

1. $s \leq M \ \forall s \in S$

2. $M \in S$

$\nexists M \in \mathbb{R}_-$ s.t. $a \leq M \ \forall a \in \mathbb{R}_-$

Note that if S has a max, then max must be the sup.

If $t \in X$ s.t. $s \leq t \ \forall s \in S$ then $M \in S$ so $M \leq t$.

But in $\mathbb{R}$, possible to have a sup but not a max

Ex. $\mathbb{R}_-$ does not have a max, but does have a sup 0.

Let $S = \{ \frac{M}{N} \mid M = 0, 1, 2 \ldots, \frac{M}{N} < b \}$

Then 1) $S \neq \emptyset$ $(0 \in S)$

2) S bounded above by b

$\sup(S) = r$

Claim. r solves our problem r is rational b/t a, b

S is a finite set. Process has to stop otherwise go above b.

Adding $1/N$ each time, go over b.

$\Rightarrow S$ is finite $\Rightarrow \sup(S) = \max(S) = r \Rightarrow r \in S$

Therefore r rational, and $r < b$

3)
Need to show $r > a$

Suppose $r = \frac{M}{N} \leq a$. Then $b - a > \frac{1}{N} \Rightarrow b > a + \frac{1}{N}$

$\geq \frac{M}{N} + \frac{1}{N} = \frac{M+1}{N}$

So $\frac{M+1}{N} < b \Rightarrow \frac{M+1}{N} \in S$

but $\frac{M+1}{N} > \frac{M}{N} = r$

So found an element in S that is bigger than r

but that contradicts the fact that $r = \max(S)$. □

Thm $\mathbb{R}$ is uncountable (there is no bijection betw $\mathbb{N} \to \mathbb{R}$)

Lets look at a simpler problem and focus on the interval $(0,1)$

Claim $\{x \in (0,1)$ is uncountable$\}$

> ★ We will assume $\{x \in (0,1)$ is countable$\}$ and derive a contradiction.
>
> if $\{x \in (0,1)$ countable$\}$, the elements can be enumerated in some form.

Proof: Cantor's Diagonalization Argument

Suppose we can list all the numbers in $(0,1)$

$a_1 = 0.a_{11}a_{12}a_{13}\ldots$

$a_2 = 0.a_{21}a_{22}a_{23}$

$a_3 = 0.a_{31}a_{32}a_{33}\ldots$

where a_{ij} are the digits

$a_{ij} \in \{0, \ldots, 9\}$

Consider the diagonal $d = 0.a_{11}a_{22}a_{33}a_{44}\ldots$

Construct $x = 0.x_1x_2x_3$ with the property that $x_i \neq a_{ii}$ and $(x_i \neq 9)$

Q: why $x_i \neq 9$? Want to avoid $0.999\bar{9} = 1.000 = 1 - 0.00\bar{0}1 = 1$

For the same reason that $1 = (0.222\ldots)_3$ in ternary $\quad 1 = \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ infinite progression

The claim is that x is not on the list of reals

$x_1 \neq a_{11}$ by def, so x cannot be a_1

$x_2 \neq a_{22}$ by def, so x cannot be a_2

x is not equal to a_i $\forall i$, therefore x is not on the list

- If $x = a_i$ for some i then the i^{th} digit of x would be equal to a_{ii} but $x_i \neq a_{ii}$ by construction.

$\Rightarrow$ Even if you counted all the reals, there will always be one missing.